Colorful homomorphisms

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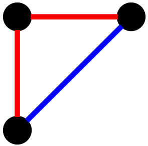
This thesis is based on joint work with Grzegorz Gutowski.

k-edge-colored graphs

A k-edge-colored graph $\mathbb{G}$ is a pair (G, c), where:

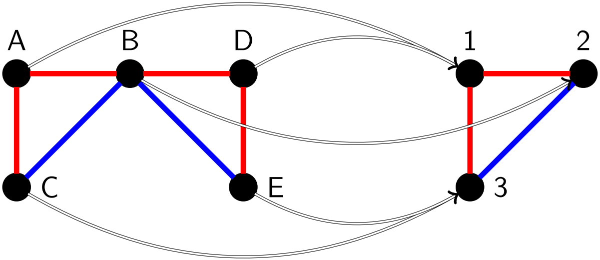
- G is a graph,
- c is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$.



Homomorphism

$\mathbb{G}, \mathbb{H}$ - two k-edge-colored graphs.

A mapping $h : V(\mathbb{G}) \to V(\mathbb{H})$ is a homomorphism if every colored edge $\{u, v\}$ maps to an edge $\{h(u), h(v)\}$ of the same color.



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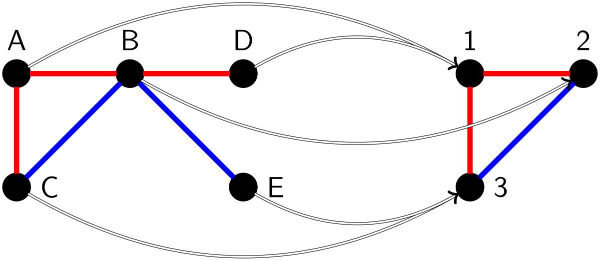

k-universal graphs

A k-edge-colored graph $\mathbb{H}$ is k-universal for a graph class $\mathcal{F}$ if every k-edge-colored graph with the underlying graph in $\mathcal{F}$ maps homomorphically to $\mathbb{H}$.

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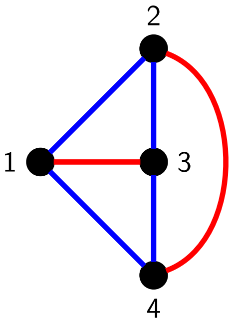

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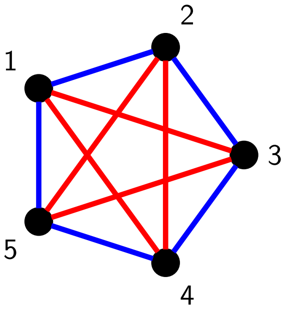

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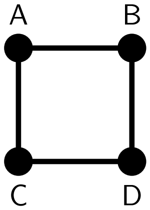

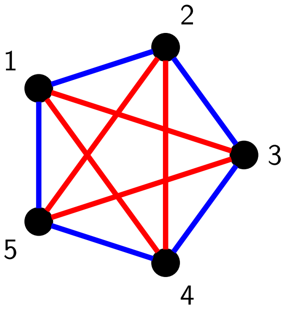

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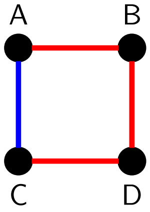

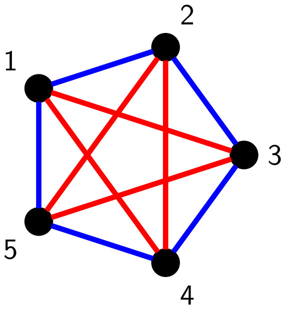

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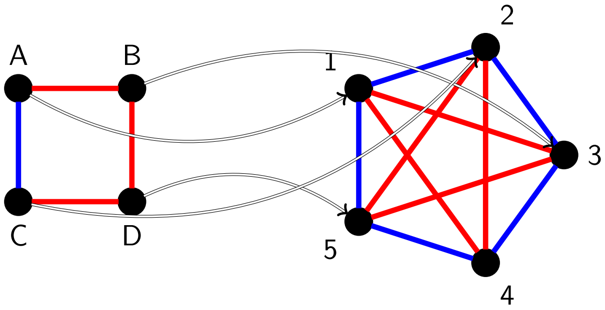

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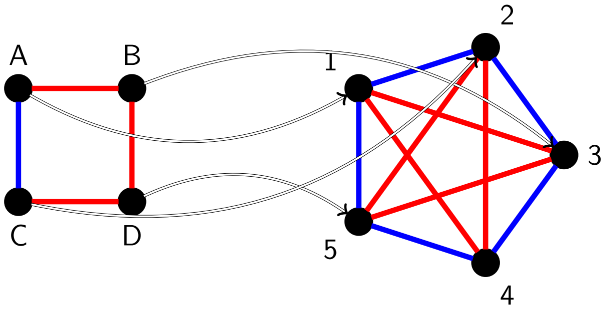

Theorem (Alon, Marshall '98)

Planar graphs admit a k-universal graph of size between $k^3 + 3$ and $5k^4$.

First theorem

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$\mathcal{F}$ admits k-universal graphs

$$\Updownarrow$$

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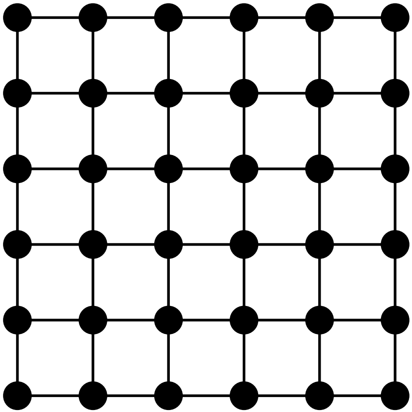
Remarks:

- "$\Uparrow$" proven by Alon and Marshall.

Acyclic chromatic number

A vertex coloring is acyclic if:

- it is a proper coloring,
- edges between every two colors form a forest.



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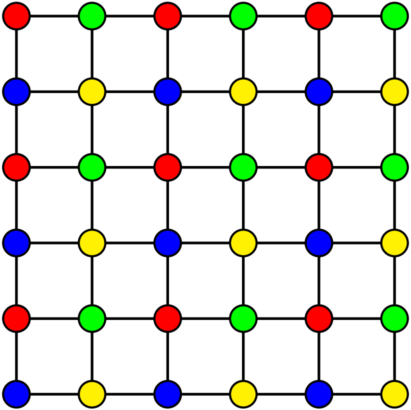

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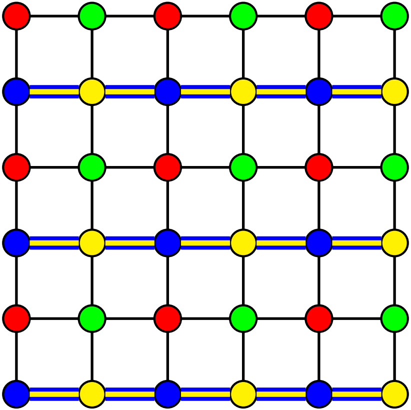

Main theorem

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$\mathcal{F}$ - a class of graphs with acyclic chromatic number bounded by a constant.
The function $\lambda_{\mathcal{F}}(k)$ satisfies:

- $\lambda_{\mathcal{F}}(k) = O(k^{\lceil D(\mathcal{F}) \rceil})$,
- $\lambda_{\mathcal{F}}(k) = \Omega(k^{D(\mathcal{F})})$,

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Density of a graph G:

$$D(G) = \max \left\{ \frac{|E(G')|}{|V(G')|} \ : \ G' \text{ is a nonempty subgraph of } G \right\}.$$

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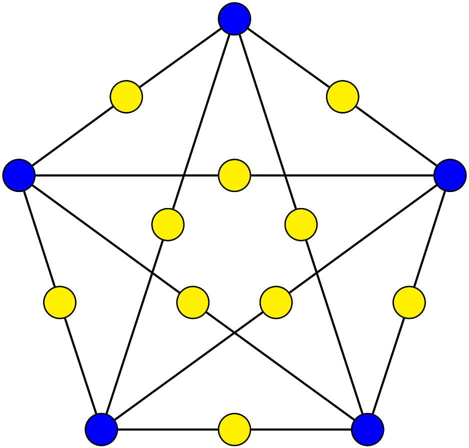
Density of a graph class $\mathcal{F}$:

$$D(\mathcal{F}) := \sup \{ D(G) : G \in \mathcal{F} \}.$$

Example: clique subdivisions

$SK_n :=$ 1-subdivision of K_n



$\mathcal{S} := \{SK_1, SK_2, \ldots\}$

Comparison to the theorem of Alon and Marshall

Theorem (Alon, Marshall '98)

$\mathcal{F}$ - a class of graphs with acyclic chromatic number bounded by r. The function $\lambda_{\mathcal{F}}(k)$ satisfies the bound:

$$\lambda_{\mathcal{F}}(k) \leqslant rk^{r-1}.$$

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Remarks:

- $D(G) \leqslant r - 1$,

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The function $\lambda_{\mathcal{F}}(k)$ satisfies the bound:

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Remarks:

- $D(G) \leqslant r - 1$,
- $\lambda_{\mathcal{F}}(k) = O(k^{\lceil D(\mathcal{F}) \rceil})$ never worse than $\lambda_{\mathcal{F}}(k) = O(k^{r-1})$.

Applications

Planar graphs $\mathcal{P}$:

- $\lambda_{\mathcal{P}}(k) = \Theta(k^3)$ as $D(\mathcal{P}) = 3$,
- $\lambda_{\mathcal{P}}(k) = O(k^4)$ as the acyclic chromatic number of any planar graph is at most 5 (Borodin '79).

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Graphs $\mathcal{G}(g)$ with genus bounded by g:

- $\lambda_{\mathcal{G}(g)}(k) = k^{\Theta(\sqrt{g})}$ as $D(\mathcal{G}(g)) = \Theta(\sqrt{g})$ by Euler formula,
- $\lambda_{\mathcal{G}(g)}(k) = k^{O\left(g^{\frac{4}{7}}\right)}$ (Alon, Mohar, Sanders '96).

Techniques

An orientation $\vec{G}$ of graph G is a d-orientation if the in-degree of every vertex is bounded by d.

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Main tools:

- Hakimi's lemma ($D(G) = d$ gives a $\lceil d \rceil$-orientation $\vec{G}$),
- an auxiliary coloring of $\vec{G}$ (from acyclic coloring).

Acknowledgements