Connecting the Dots (with Minimum Crossings)

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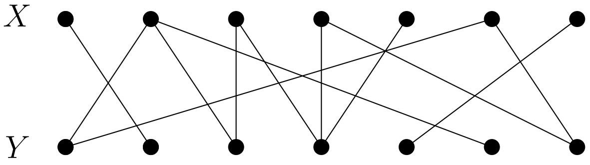
Saket Saurabh, Meirav Zehavi

21 June 2019

Crossing-Minimizing Perfect Matching:

Input:
- a bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$,
- linear orders $<_X$ of X and $<_Y$ of Y,
- a nonnegative integer k.

Question: Does G admit a perfect matching $M \subseteq E$ with at most k crossings?



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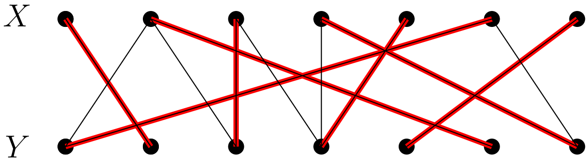

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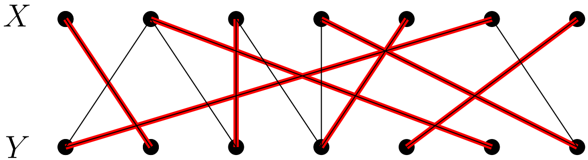
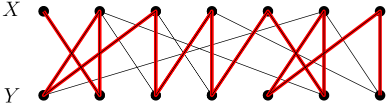
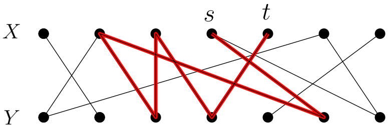




Hamiltonian path



(s, t)-path

Parametrized complexity

Parametrized problems

decision problem $\xrightarrow[\text{with each instance}]{\text{associate an integer } k \text{ (parameter)}}$ parametrized problem

FPT algorithms

$f(k)n^{\mathcal{O}(1)}$ – fixed parameter tractable (FPT)

Kernelization

instance (I,k) of size n $\xrightarrow[\text{(kernelization algorithm)}]{\text{algorithm running in time } n^{O(1)}}$ equivalent instance of size $f(k)$ (kernel)

Hardness in parametrized complexity

W-hardness

Our results

	perfect matching	Hamiltonian path	(s,t)-path
NP-complete?	**yes**, even on graphs of maximum degree 2	**yes**, even on graphs that admit a Hamiltonian path	**yes**
lower bounds	✗ $2^{o(n+m)}$ ✗ $2^{o(\sqrt{k})}n^{\mathcal{O}(1)}$ (under ETH)	✗ $2^{o(n+m)}$ ✗ $2^{o(\sqrt{k})}n^{\mathcal{O}(1)}$ (under ETH)	W[1]-hard
parametrized algorithm	$2^{\mathcal{O}(\sqrt{k})}n^{\mathcal{O}(1)}$	$2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$	$n^{\mathcal{O}(k)}$ (XP)
kernel	$\mathcal{O}(k^2)$ vertices	$\mathcal{O}(k^2)$ vertices	—

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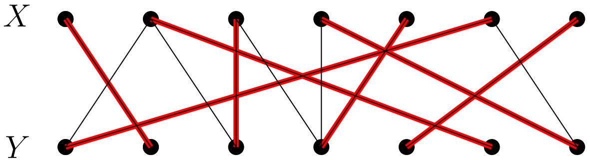
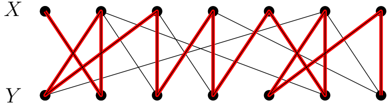
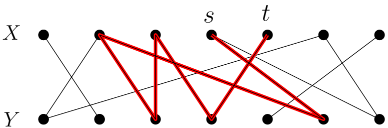




Hamiltonian path



(s, t)-path

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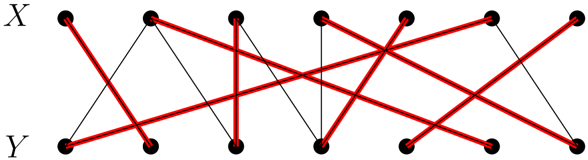
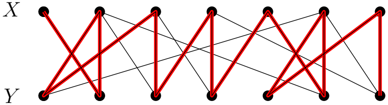
Parameter: k.





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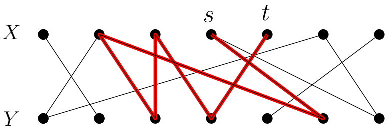

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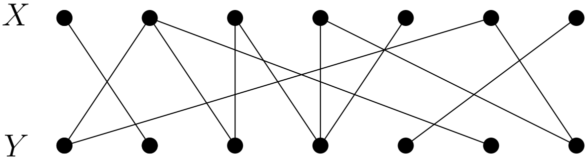
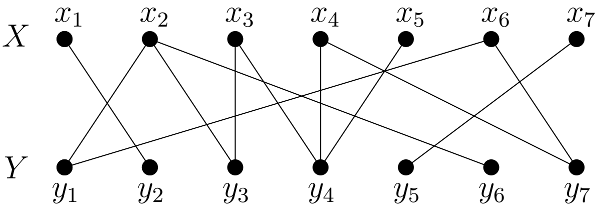
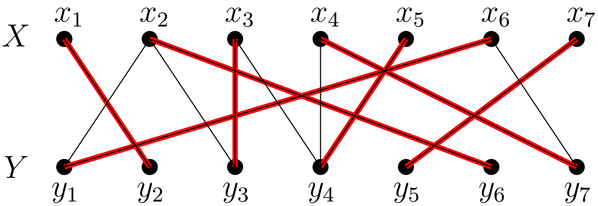
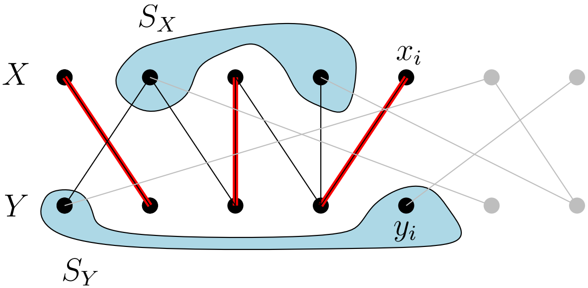

X
Y


X
Y
x1 x2 x3 x4 x5 x6 x7
y1 y2 y3 y4 y5 y6 y7

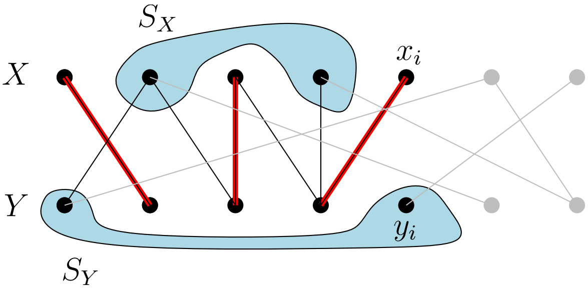


X
x_1
x_2
x_3
x_4
x_5
x_6
x_7
Y
y_1
y_2
y_3
y_4
y_5
y_6
y_7



- $t[i][S_X][S_Y]$ – minimum number of crossings in a matching in G that matches the vertices $\{x_1, \ldots, x_i\} \setminus S_X$ with the vertices $\{y_1, \ldots, y_i\} \setminus S_Y$ (∞ if no such matching),

  for every $i = 1, \ldots, n$, $S_X \subseteq \{x_1, \ldots, x_i\}$, and $S_Y \subseteq \{y_1, \ldots, y_i\}$.



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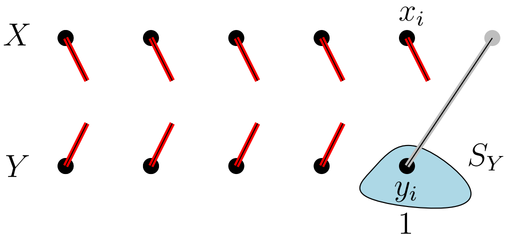
- 4^i subproblems for each i – too many.



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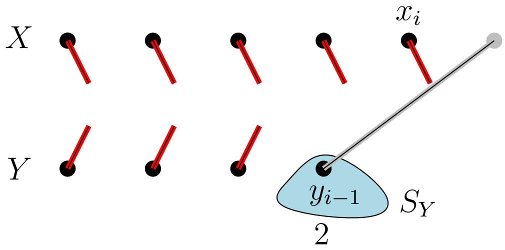

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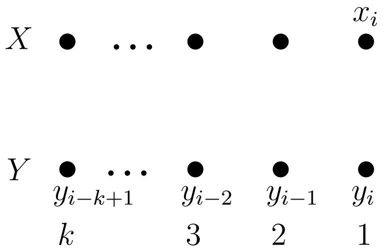

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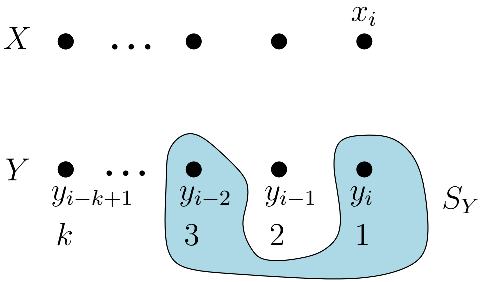

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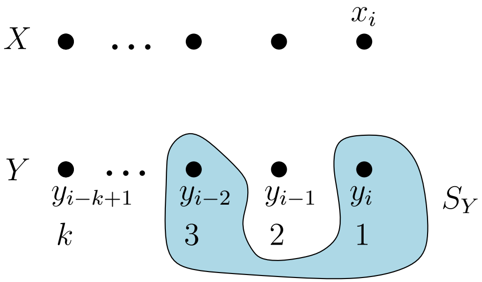

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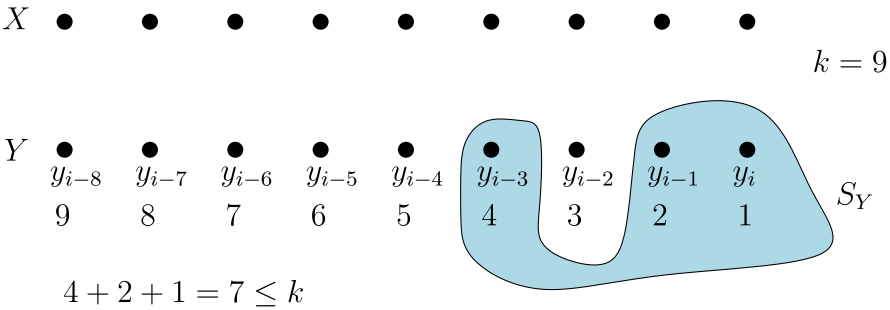

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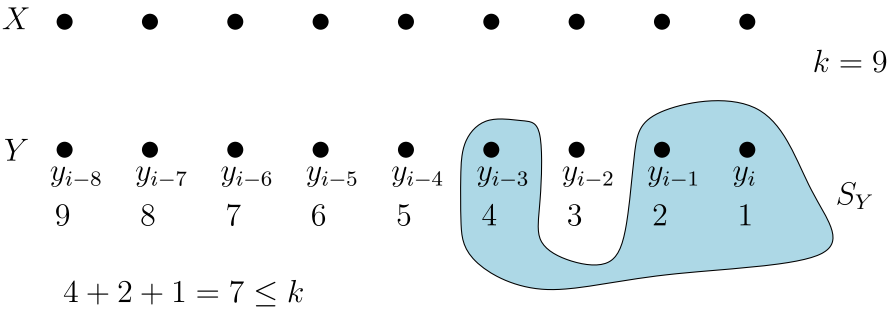

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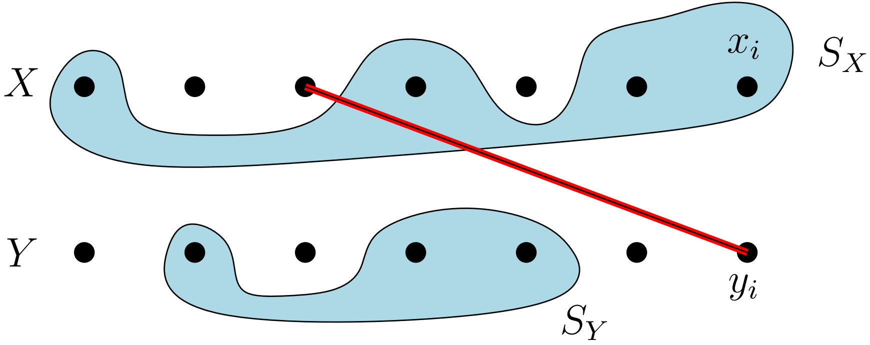
- At most $2^{\mathcal{O}(\sqrt{k})}$ subproblems for a given i, and $n2^{\mathcal{O}(\sqrt{k})}$ in total.

Case 1: $x_i \in S_X, y_i \notin S_y$



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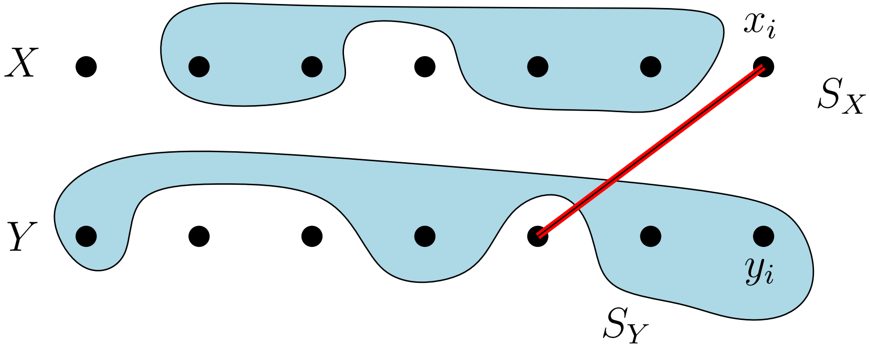
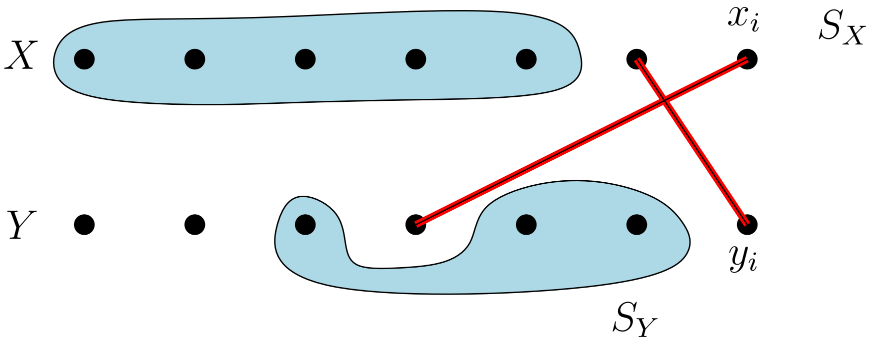
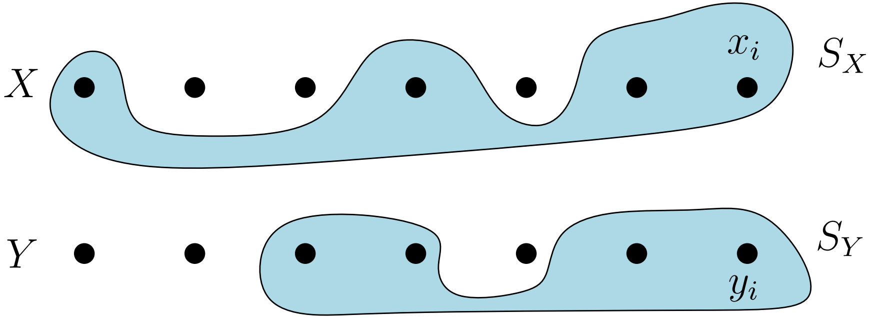


Case 3: $x_i \notin S_X, y_i \notin S_y$



Case 4: $x_i \in S_X, y_i \in S_y$



Thank you!