Smaller Universal Targets for Homomorphisms of Edge-colored Graphs

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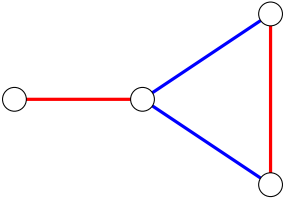
西安, 30 July 2019

A k-edge-colored graph $\mathbb{G}$ is a pair (G, c), where:

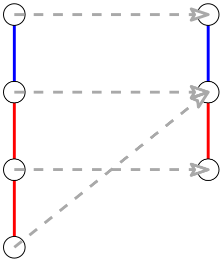
- G is a graph,
- c is a mapping from $E(G)$ to $\{1, \ldots, k\}$.



Homomorphism

$h : V(\mathbb{G}) \to V(\mathbb{H})$ is a homomorphism if for every edge $uv \in E(\mathbb{G})$:

- $h(u)h(v) \in E(\mathbb{H})$,
- $c_{\mathbb{H}}(h(u)h(v)) = c_{\mathbb{G}}(uv)$.



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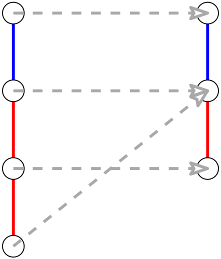

Universal graph

$\mathbb{H}$ is *k*-universal for a class $\mathcal{F}$ of graphs if every k-edge-coloring of every graph in $\mathcal{F}$ maps homomorphically to $\mathbb{H}$.

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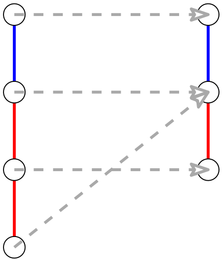

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$k^3 + 3 \leqslant \lambda_{\text{PLANAR}}(k) \leqslant 5k^4$.

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Theorem (Guśpiel, Gutowski '15)

$\lambda_{\text{PLANAR}}(k) \leqslant 8435812575000000 \cdot k^3.$

Acyclic coloring

- Every two adjacent vertices get different colors.
- Vertices of any cycle in the graph get at least 3 different colors.

$\chi_a(G)$, $\chi_a(\mathcal{F})$.

Theorem (Guśpiel, Gutowski '15)

$\lambda_{\mathcal{F}}(k) < \infty \iff \chi_a(\mathcal{F}) < \infty.$

(' $\Longleftarrow$ ' by Alon, Marshall '98)

Density

$$D(G) = \max\left\{\frac{|E(G')|}{|V(G')|} \ : \ G' \text{ is a nonempty subgraph of } G\right\}.$$

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If $\chi_a(\mathcal{F}) < \infty$, *then*

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Theorem

If $\chi_a(\mathcal{F}) < \infty$ and $D(\mathcal{F})$ is a rational number, then

$$k^{D(\mathcal{F})} \leqslant \lambda_{\mathcal{F}}(k) \leqslant C'k^{D(\mathcal{F})}.$$

How to get $O\left(k^{\lceil D(\mathcal{F})\rceil}\right)$?

k edge colors, $r = \chi_a(\mathcal{F}), d = \lceil D(\mathcal{F})\rceil$

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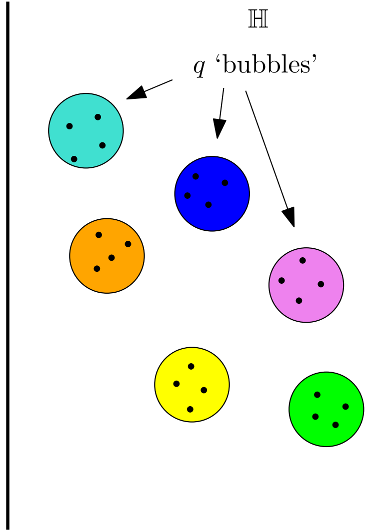

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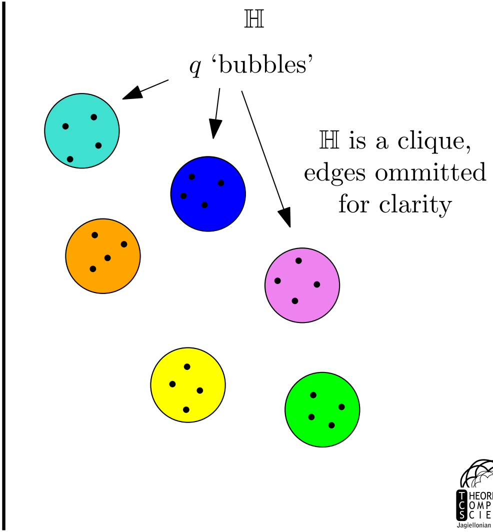

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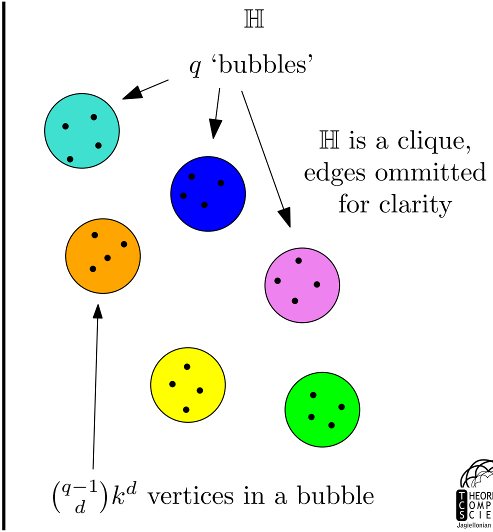

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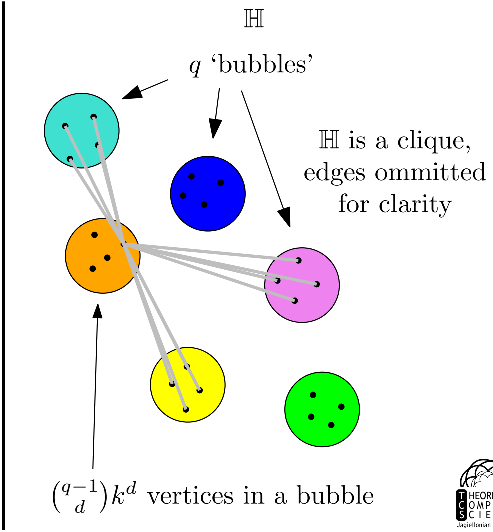

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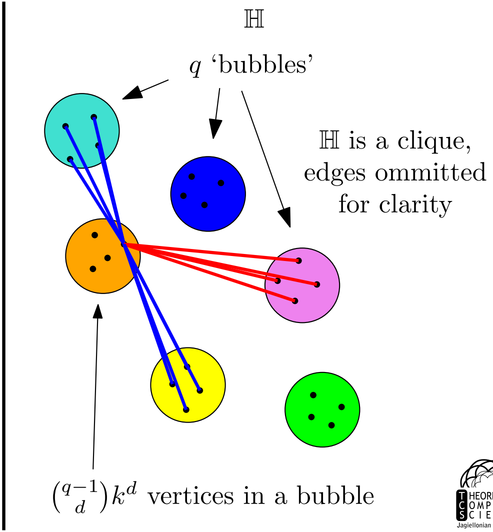

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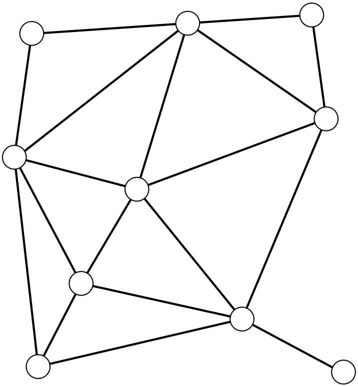
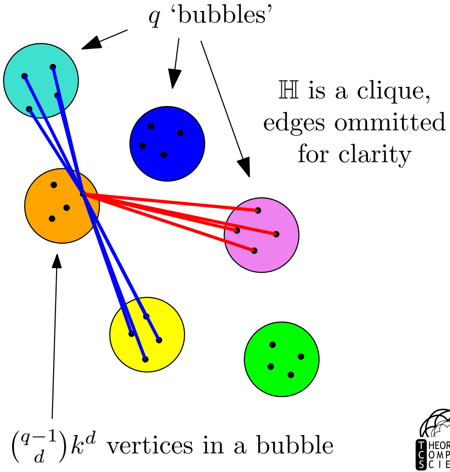
$G \in \mathcal{F}$



$\mathbb{H}$



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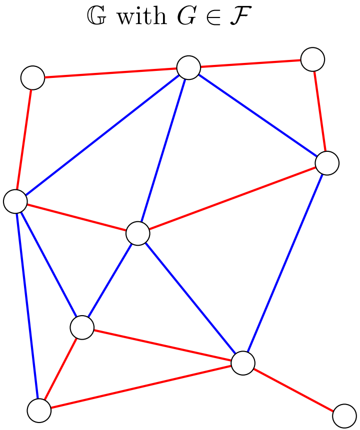

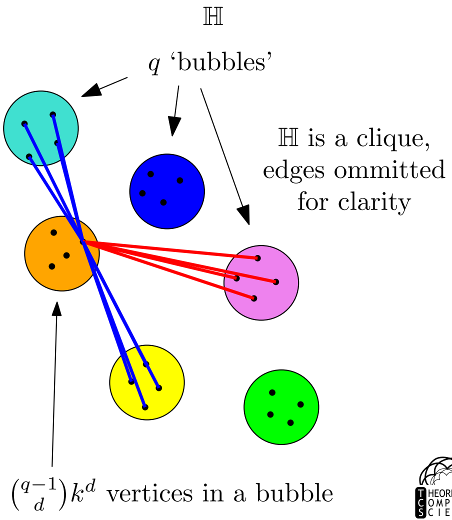

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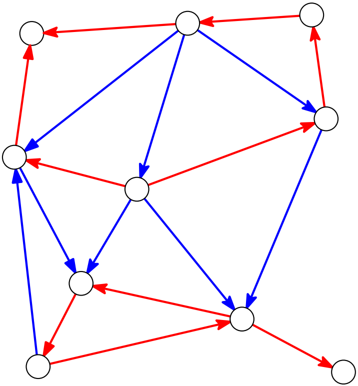

orientation of $\mathbb{G}$ with indeg $\leq d$

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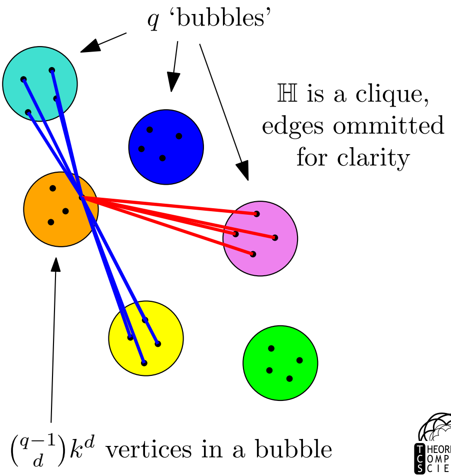

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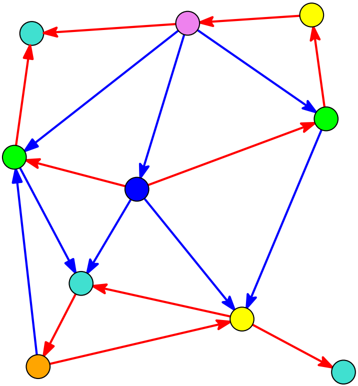
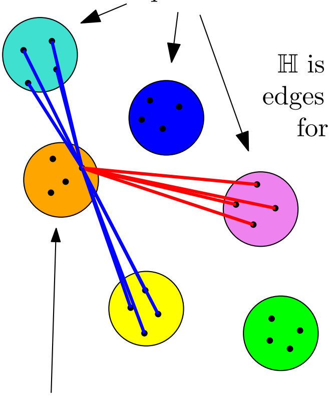


orientation of $\mathbb{G}$ with indeg $\leq d$
special q-coloring of $V(G)$

$\mathbb{H}$

q ‘bubbles’



$\mathbb{H}$ is a clique, edges ommitted for clarity

$\binom{q-1}{d} k^d$ vertices in a bubble

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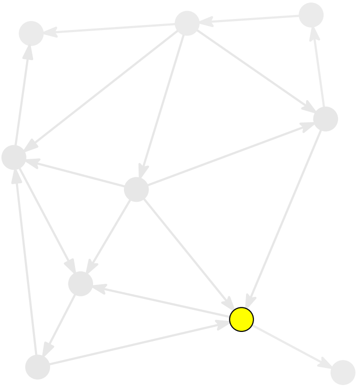

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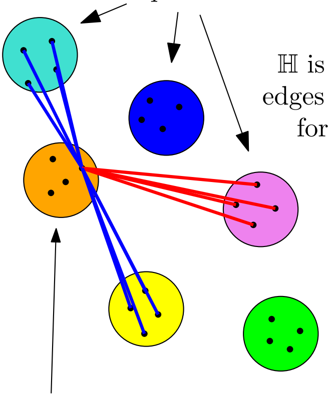

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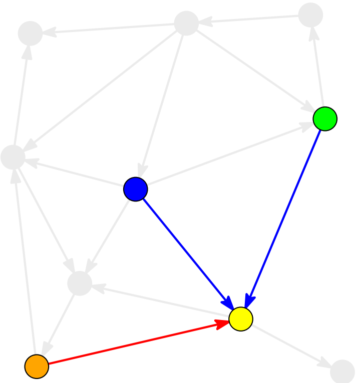

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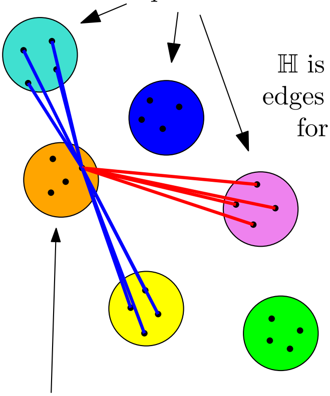

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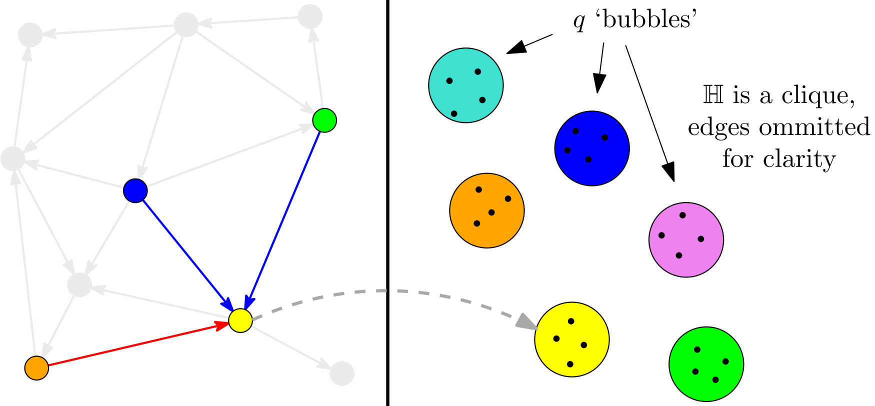

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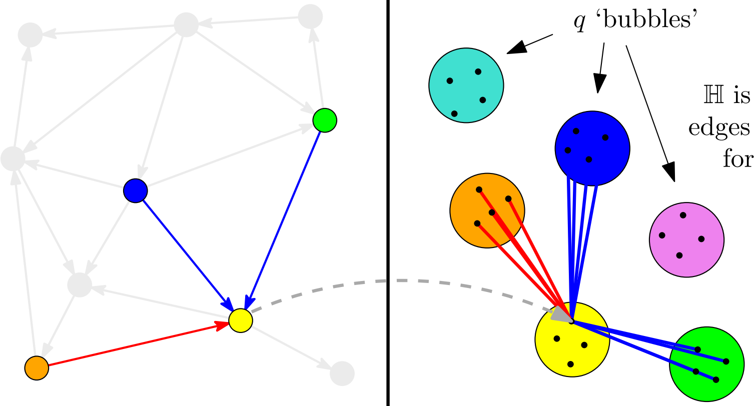

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k edge colors, $\chi_a(\mathcal{F}) = r$, $D(\mathcal{F}) = s/t$

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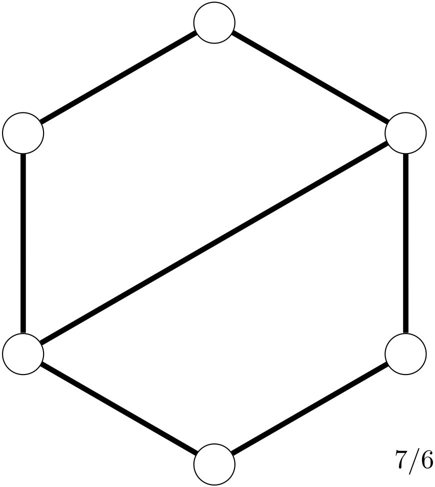

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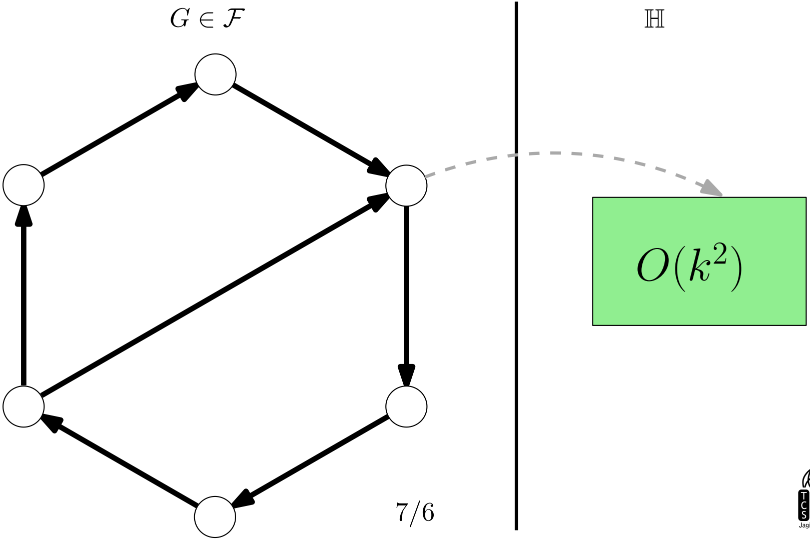

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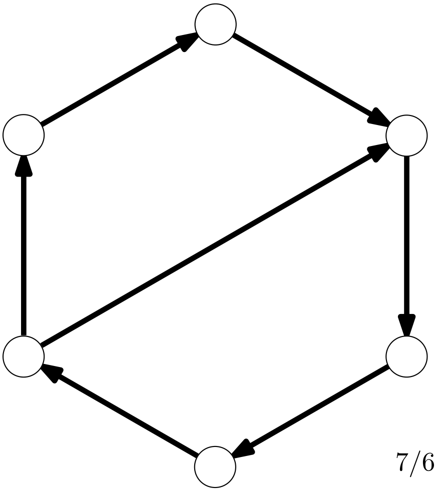

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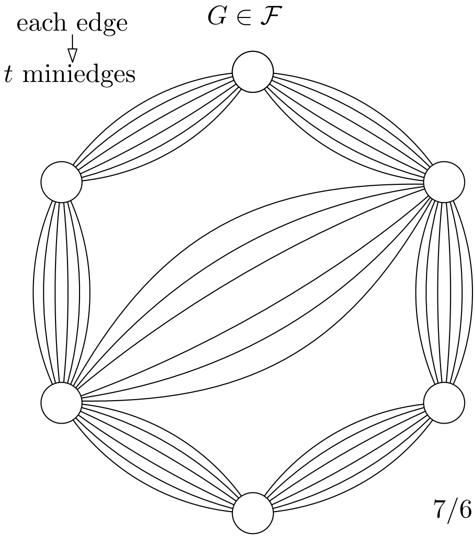

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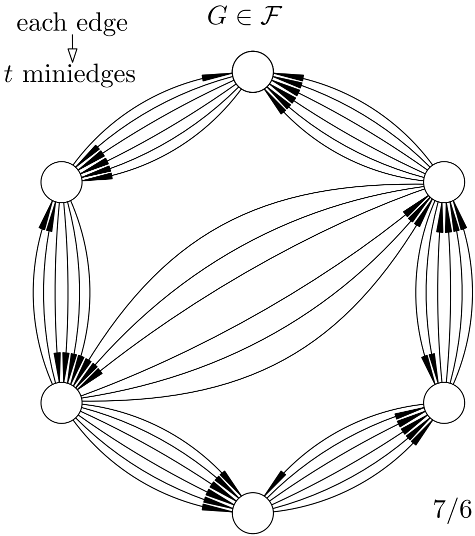

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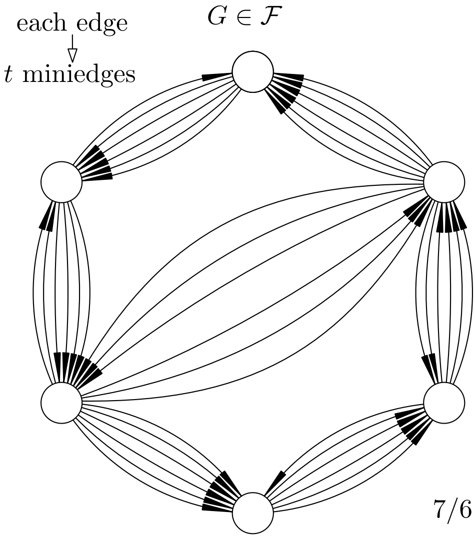

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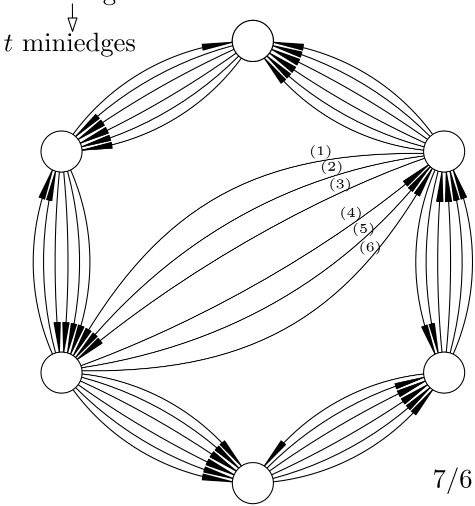

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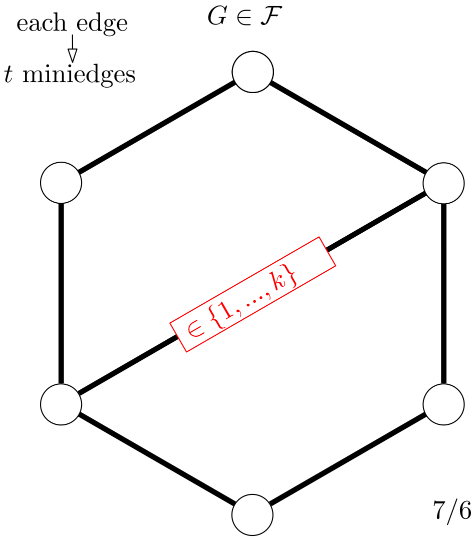

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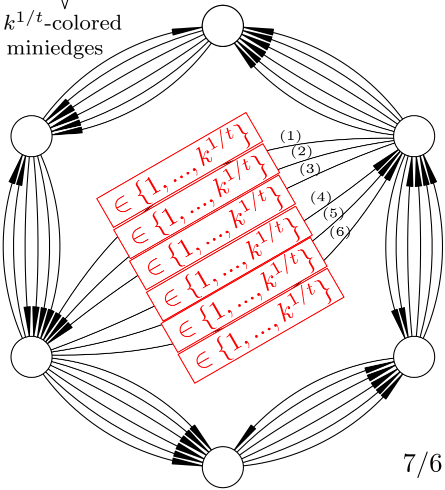
each k-colored edge

↓

t $k^{1/t}$-colored miniedges

$G \in \mathcal{F}$



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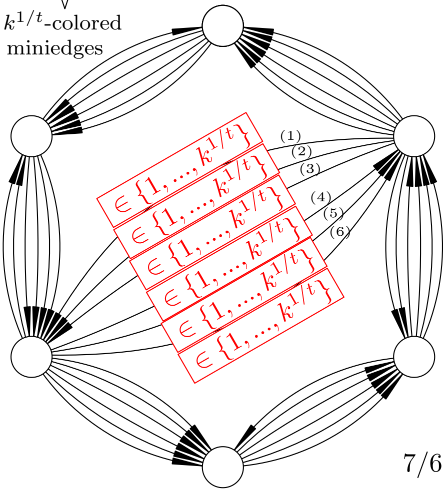

$\mathbb{H}$

(t $k^{1/t}$-colored miniedges between each pair of vertices)

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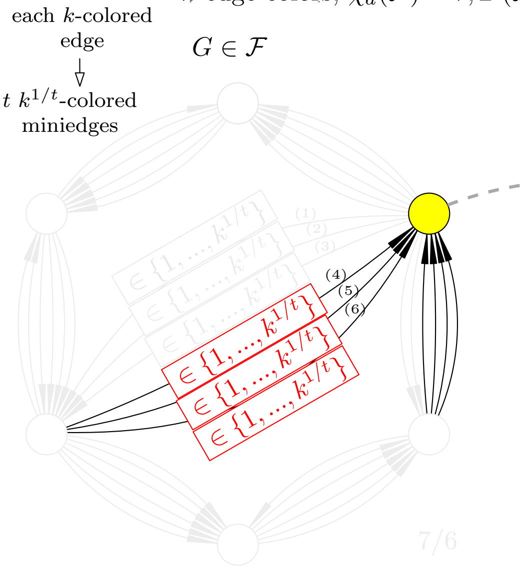
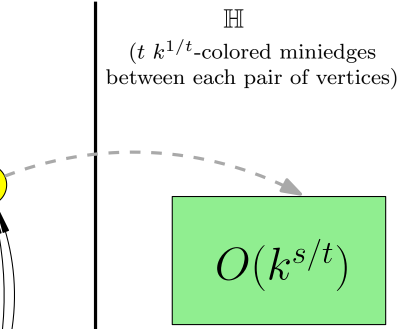
k edge colors, $\chi_a(\mathcal{F}) = r$, $D(\mathcal{F}) = s/t$





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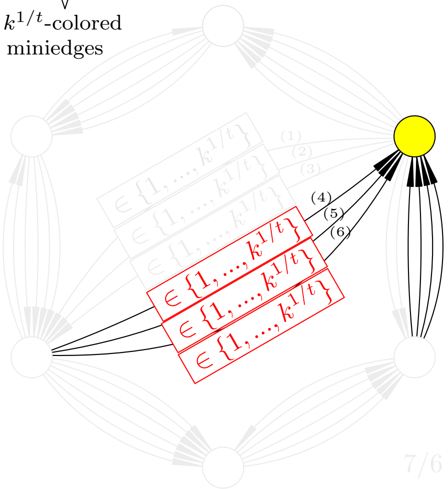
How to get $O\left(k^{D(\mathcal{F})}\right)$ for $D(\mathcal{F}) = s/t$?

k edge colors, $\chi_a(\mathcal{F}) = r$, $D(\mathcal{F}) = s/t$

each k-colored edge
↓
t $k^{1/t}$-colored miniedges

$G \in \mathcal{F}$



$\mathbb{H}$

(t $k^{1/t}$-colored miniedges between each pair of vertices)

$\mathbb{H}_0$ – $k^{1/t}$-universal graph

$$\mathbb{H}_0 \times \mathbb{H}_0 \times \ldots \times \mathbb{H}_0$$

$$\Cup$$

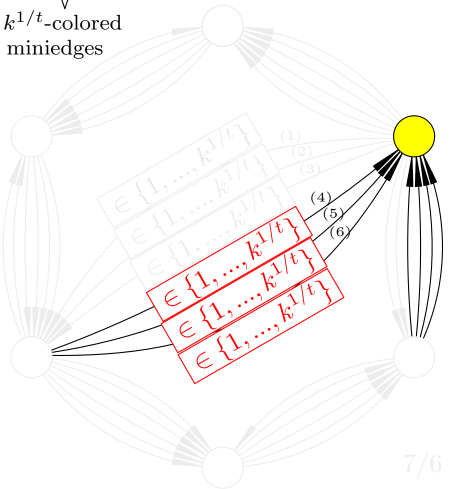
$$(v_1, v_2, \ldots, v_t)$$

How to get $O\left(k^{D(\mathcal{F})}\right)$ for $D(\mathcal{F}) = s/t$?

k edge colors, $\chi_a(\mathcal{F}) = r$, $D(\mathcal{F}) = s/t$

each k-colored edge
$\downarrow$
t $k^{1/t}$-colored miniedges

$G \in \mathcal{F}$



$\mathbb{H}$

(t $k^{1/t}$-colored miniedges between each pair of vertices)

$\mathbb{H}_0$ – $k^{1/t}$-universal graph

$$\mathbb{H}_0 \times \mathbb{H}_0 \times \ldots \times \mathbb{H}_0$$

$$\Downarrow$$

$$(v_1, v_2, \ldots, v_t)$$

$$\ldots$$

$$(u_1, u_2, \ldots, u_t)$$

Thank you!